Assignment 1

Hand in no. 1, 3b, 4, 5 by September 19.

1. Let f be a 2π-periodic function which is integrable over $[-\pi, \pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx = \int_J f(x)dx,$$
where I and J are intervals of length 2π.

2. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

3. Here all functions are defined on $[-\pi, \pi]$. (a) Sketch their graphs as 2π-periodic functions, (b) find their Fourier series and (c) determine the convergence and uniform convergence of these Fourier series (if possible).

   (a)
$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)
$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

   (c)
$$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

   (d)
$$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi] \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

4. Let f be a 2π-periodic function whose derivative exists and is integrable on $[-\pi, \pi]$. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebesgue Lemma. Hint: Use integration by parts to relate the Fourier coefficients of f to those of f'.

5. Let f be a continuous 2π-periodic function. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebsegue lemma. Hint: Establish the formula
$$2\pi a_n = \int_{-\pi}^{\pi} [f(y) - f(y + \pi/n)] \cos ny \, dy \ ,$$
using Problem 1.

6. Let g be an integrable T-periodic function. Show that for any integrable function f on $[a, b]$,
$$\lim_{n\to\infty} \int_a^b f(x)g(nx)\, dx = \frac{1}{T}\int_0^T g(x)\, dx \int_a^b f(x)\, dx \ .$$
Suggestion: Consider the special case $\int_0^T g(x)dx = 0$ first.

7. A sequence of integrable functions $\{g_n\}_1^\infty$ on $[a, b]$ is called an orthonormal family if (a) $\int_a^b g_n(x)g_m(x)\,dx = 0$ for $n \neq m$ and $\int_a^b g_n^2(x)\,dx = 1$ for all n. Show that whenever $f(x) = \sum_{n=1}^\infty c_n g_n(x)$ holds, $c_n = \int_a^b f(x)g_n(x)\,dx$ for all n. Is the family $\{1, \cos nx, \sin nx\}$ orthonormal on $[-\pi, \pi]$?